Reinforcing the resilience of complex networks

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Given a connected network, it can be augmented by applying a growing strategy (e.g., random- or preferential-attachment rules) over the previously existing structure. Another approach for augmentation, recently introduced, involves incorporating a direct edge between any two nodes which are found to be connected through at least one self-avoiding path of length L. This work investigates the resilience of random- and preferential-attachment models augmented by using the three schemes identified above. Considering random- and preferential-attachment networks, their giant cluster are identified and reinforced, then the resilience of the resulting networks with respect to highest-degree node attack is quantified through simulations. Statistical characterization of the effects of augmentations over some of the network properties is also provided. The results, which indicate that substantial reinforcement of the resilience of complex networks can be achieved by the expansions, also confirm the superior robustness of the random expansion. An important obtained result is that the initial growth scheme was found to have little effect over the possibilities of further enhancement of the network by subsequent reinforcement schemes.

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I. INTRODUCTION

Much has been learnt about several aspects of complex networks [1–3] by looking at such models from different theoretical and practical points of view, such as network growth and critical phenomena (e.g., [3,4]), node degree distribution (e.g., [1]), distance between nodes (e.g., [5]), diffusion (e.g., [6]), and resilience to attack, to name but a few. As each of these situations drives the researcher to focus attention on specific topological and functional aspects of the investigated networks, they contribute to a more comprehensive and integrated understanding of the many complexities of networks. The current work addresses the resilience issue by taking into account the following three important perspectives. First, we target the situation where one wants to enhance an already existing network with respect to attacks by adding new edges; second, we consider the abrupt change of rules during the network growth, producing *hybrid* models; and third, we investigate the potential of the recently introduced concept of L expansion of a network [7] for enhancing resilience.

Because of its immediate practical consequences to Internet and distributed systems, the problem of characterizing the resilience of complex networks has received growing attention, especially after the seminal papers by Albert *et al.* [8], who addressed node deletion in scale-free models of Internet, and investigation by Callaway *et al.* [9] on random networks under attack. Other related works include the [10] comprehensive comparative investigation by Holme *et al.* [10] of the resilience of several types of networks considering different schemes for attacking nodes and edges, and the analysis of internet breakdown [11] by Cohen *et al.* Interesting works on network performance under attack have also appeared in computer science journals, including the in-depth investigation of Internet under attack at the network level [12] and the simulation approach to denial of service attacks [13], as well as the studies of internet topology and fault tolerance reported in Ref. [14]. Works targeting specific types of network include, but are not limited to, Newman's investigation of e-mail networks [15], study of metabolic systems [16] by Jeong *et al.*, and Dunne's analysis of food webs [17]. More recently, the concept of L expansions of a complex network was suggested [7] which, by enhancing the network connectivity, was believed to present good potential for increasing the resilience of existing networks.

This paper starts by reviewing the concept of L expansions and augmentations of a network and follows by discussing hybrid networks. Statistical analysis and predictions about the effect of the augmentations over the network properties are presented next, followed by the discussion of the obtained results and identification of perspectives for further investigation.

II. Q AUGMENTATIONS OF A NETWORK

Recently introduced [7], the concept of L expansion of a given network (any type, directed or not) seems to provide good potential for reinforcing the connectivity regularity of existing networks, with implications for resilience. Given a graph Γ, its L expansion consists of a graph where connections from node i to j are established whenever there exists a self-avoiding path (i.e., never passing by the same node twice) of length L connecting i to j in Γ. Here we introduce the concept of Q-augmented network in order to express networks obtained by the union of the original graph with its respective L-*expanded* models for $L=2$ to $L=Q$. This simple concept is illustrated in Fig. 1, which shows an original network (a) and its respective two-, three-, and four-augmented versions. It is interesting to observe that these augmentations reinforce the regularity of the network up to length Q. An important global measurement of the effect of the augmen-

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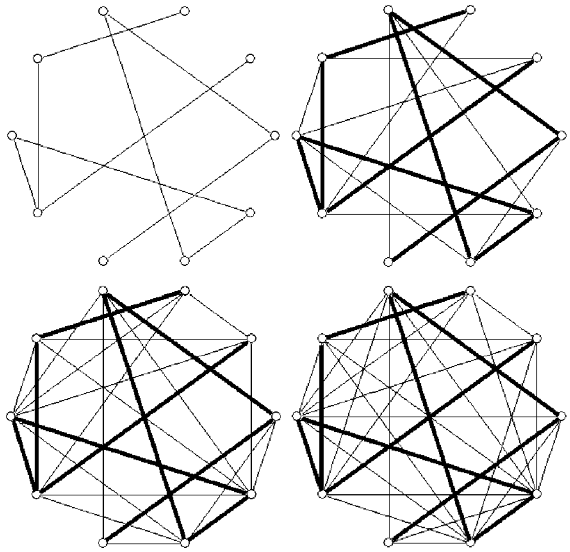

FIG. 1. A simple graph (a) and its respective two- (b), three- (c), and four-augmentations (d). The edges of the initial network (a) are shown by thicker lines in (b-d).

tation on the network connectivity is the ratio between the number of connections in the augmented and original networks, henceforth represented as ρ and denominated *augmentation ratio*. This work is restricted to $Q=3$ and 4, as smaller values lead to rather uneffective resilience reinforcement and higher values would imply extreme augmentation of the network connectivity.

III. HYBRID NETWORKS

Hybrid networks are here understood as graphs obtained through the application of more than one growing strategy. Although such a mixup of evolutions can take place simultaneously, here we consider the situation where the network was grown under a scheme which was subsequently switched to another growing strategy. Such alternation of schemes, to several degrees of switching abruptness, are typical of several real networks. For instance, the neuronal networks constituting the central nervous systems of mammals are known to undergo major changes of topology and connecting rules during the first weeks of life of the individual as a consequence of the exposition of such networks to dynamical stimuli [18,19]. In other words, the animals are born with pre-wired networks that are subsequently rewired and pruned down as a consequence of the presentation of real stimuli. Other situations where the growing network undergoes abrupt changes of connectivity regard, for instance, the introduction of new technologies. The existing telephonic network, for instance, includes older physical connections through cables augmented by the incorporation of satellite and cellular telephony (e.g., Refs. [20,21]). In this case, while the previous network exhibited a more regular nature imposed by the spatial constraints of wiring cables (spatial adjacencies), the satellites and antennae incorporated at later stages concentrate connections, acting as the hubs of scale-free networks. A similar situation occurred when air transportation (e.g., Ref. [22]) was introduced in addition to trains and automobiles, with the main airports acting as hubs of a previously regular network, with the range of connections limited by the two-dimensional adjacencies underlying car and train transportation (e.g., Ref. [23]).

The N nodes of the network of interest Γ are henceforth represented as k and the E edges as ordered pairs (i,j), with the respective adjacency matrix being expressed as A. No self-connections are allowed. Given a network Γ_α of a specific type α (e.g., random- or preferential-attachment), its augmentation (see, e.g., Ref. [24]) $\Pi_\alpha(\Gamma_\beta)$ can be obtained by applying the growing rules of any other model type β *over the existing network* Γ, implying the addition of ΔE new edges but without changing the number of nodes. It is observed that the term *preferential attachment* has been used instead of *scale free* as it will not typically lead to a scale-free network when used as expansion model. Among the several subclasses of scale-free networks (e.g., Refs. [25,26], the preferential attachment growing scheme used in the current work involves choosing a pair of nodes for connection with probability proportional to the respective number of connections (or node degrees). More specifically, a list of the nodes participating in connections is kept at all times, with repetitions, and the node candidates for connections are drawn from such a list with uniform probability (e.g., Ref. [27]). The random model involves selecting from among the $N(N-1)/2$ possible connections according to the uniform statistical distribution (e.g., Ref. [1]).

Therefore, ΔE corresponds to the number of edges that are added to the network as a consequence of the second growth stage in order to boost its resilience to attack. Thus, we can have a random model augmented by the preferential attachment, or a preferential-attachment network followed by a Q augmentation. Such combinations of growing schemes are henceforth called *hybrid augmentation*, of which the current work considers the six following situations: (i) random followed by random — i.e., $\Pi_R(\Gamma_R)$; (ii) random followed by preferential-attachment — i.e., $\Pi_{PA}(\Gamma_R)$; (iii) random followed by its Q augmentation — i.e., $\Pi_{QE}(\Gamma_R)$; (iv) Preferential attachment followed by random — i.e., $\Pi_R(\Gamma_{PA})$; (v) Preferential attachment followed by preferential attachment — i.e., $\Pi_{PA}(\Gamma_{PA})$; (vi) Preferential attachment followed by its Q augmentation — i.e., $\Pi_{QE}(\Gamma_{PA})$.

Observe that the two cases where a model is followed by an augmentation of the same type are equivalent to considering a single network of the same type containing the same number of nodes and edges as in the other cases. It is interesting to observe that the augmentation of a network where $\alpha \neq \beta$ typically is *not* commutative, i.e., generally $\Pi_\alpha(\Gamma_\beta) \neq \Pi_\beta(\Gamma_\alpha)$. For the sake of a fair comparison of the models, all networks derived from the initial connected graph Γ have as approximately as possible the same number of nodes and edges. More specifically, the procedure for generating the hybrid augmented models starts by growing a network Γ_α of type α and containing N_0 nodes, and the giant cluster is identified with size $N_{GC} \leqslant N_0$ nodes and E_{GC} edges. In the case of the random model, the density of edges of the network Γ_α from which the giant cluster is extracted is determined as $\lambda_i = i/N_0$, so that $i=1,2,3$ controls the density of connections. Observe that the network percolation takes place near $i=1$. In the preferential-attachment case, the number of edges of the network from which the giant cluster is extracted is determined as $E_0=\lambda_i(N_0-1)/2$. The respectively extracted giant cluster acts as the original network Γ, which is subsequently augmented by ΔE new edges according to the model β. The augmentation by the random and preferential-attachment schemes is done so as to ensure the same number of nodes and connections for each considered situation, which is done by using the above specified values λ_i and E_0 for the random and preferential-attachment schemes, respectively. Thus the resulting network contains N_{GC} nodes and $E=E_{GC}+\Delta E$ edges, so that $\rho=E/E_{GC}=1+\Delta E/E_{GC}$. The number of edges does vary for $Q=3$ and $Q=4$, so that these two situations cannot be direcly compared.

In order to substantially reduce the dispersion of the number of nodes and edges in the initial giant cluster, only those clusters with size larger than the respective averages (which were previously identified by experimental means) were considered for augmentation. The resilience of the hybrid models was quantified by considering the giant cluster of size $M(n)$ obtained after removing an increasing number n of nodes. Although some analyses were performed with edge removal (see Sec. V), the present work concentrates on the highest-degree node removal. All simulations adopted $N_0=50$ and were carried out for 200 realizations of each configuration.

IV. STATISTICAL ANALYSIS

First we turn our attention to a mean-field analysis of the effect of the augmentation scheme over the increase of the number of edges. Let the node original network have $N=N_{GC}$ nodes and the probability of existence of an edge (i,j), $i\neq j$, be p, so that the maximum number of edges is $E_T=N(N-1)/2$, the average number of edges is $E=pE_T$ and the average node degree is $k=2E/N=p(N-1)$. For simplicity's sake, we assume that the network is sparse in the sense of including few cycles, which is a good approximation for most of the edge densities considered in this paper. We also assume the probability of having a connection between a pair of nodes to be independent of that of other edges, which largely constrains the subsequent analysis to the random model.

For $L=2$, the two expansions can only take place between subsequent pairs of edges, as illustrated in Fig. 2(a), implying a maximum of $k(k-1)/2$ new expanded edges. As p edges will already exist in the average, the effective number of new edges added by the respective expansion for each node is $(1-p)k(k-1)/2$, yielding the addition of ΔE_2 edges as estimated by Eq. (1). For $L=3$, the expansion will only take place in situations such as that depicted in Fig. 2(b), i.e., each new edge will be added between one of the nodes connected to i and one of the nodes connected to j. The maximum number of such edges therefore is $c2=k^2-k$ [28], implying the average number of new edges to be given by Eq. (2) and an average augmentation ratio of ρ_3 estimated by Eq. (4). The situation for $L=4$ similarly leads to ΔE_4 as given in Eq. (3) and the average augmentation ratio ρ_4 which can be

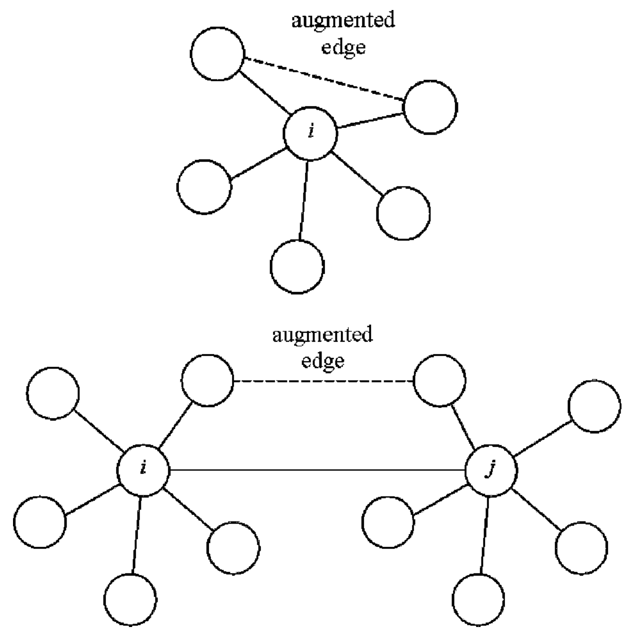

FIG. 2. The basic constructions used in the mean-field approach to estimating the augmentation ratio for expansions by $L=2$ (a) and 3 (b).

estimated by Eq. (5). It follows from this analysis, under the specified assumptions, that the number of edges added by the augmentations is, in the average, proportional to the square of the average node degree. Interestingly, as the two expansion completes all connections between the nodes attached to each network node — see Fig. 2(a), it automatically implies the clustering coefficient of the resulting Q augmented network to be one for $Q\geqslant 2$:

$$\Delta E_2 = N(1-p)k(k-1)/2, \tag{1}$$

$$\Delta E_3 = p(1-p)c^2E_T, \tag{2}$$

$$\Delta E_4 = p^2(1-p)c^2E_T, \tag{3}$$

$$\rho_3 = (E+\Delta E_2+\Delta E_3)/E, \tag{4}$$

$$\rho_4 = (E+\Delta E_2+\Delta E_3+\Delta E_4)/E. \tag{5}$$

Next we turn our attention to the prediction of the effect of the agumentation schemes over the connections of the networks. The random case immediately implies a uniform increase of the average node degree irrespective of the initial growing model. In other words, the ΔE new edges are uniformly distributed among the existing nodes by the random augmentation. On the other hand, the preferential attachment will tend to increase the node degree of those nodes whose degree was already large, according to the "rich get richer" scheme, implying a good deal of the reinforcement connections will be expended with the nodes with highest degrees. As discussed above, the addition of new connections to a node i provided by the Q augmentation is governed by the degrees of that node and those to which it connects. Two situations have to be analyzed: with respect to the reinforce-

TABLE I. The average and standard deviation values of the number of nodes, total of edges, and average node degree of the giant clusters obtained by the random- and preferential-attachment models.

		$i=1$	$i=2$	$i=3$
Random	N_{GC}	16.9 ± 2.1	41.9 ± 1.6	47.4 ± 0.4
	E_{GC}	16.2 ± 2.2	47.9 ± 1.6	72.8 ± 0.5
	Av. deg.	1.9 ± 0.1	2.3 ± 0.1	3.1 ± 0.1
Preferential-Attachment	N_{GC}	17.3 ± 2.1	40.1 ± 0.9	45.8 ± 0.7
	E_{GC}	14.0 ± 2.3	48.1 ± 0.8	72.8 ± 0.4
	Av. degree	1.9 ± 0.1	2.4 ± 0.6	3.2 ± 0.7

TABLE II. The average and standard deviation values of the experimental augmentation ratio ρ and the respective theoretical predictions for the six hybrid models considered in this work (Theor. = theoretical and Exper. = experimental).

	Q	Type	$i=1$	$i=2$	$i=3$
Random	3	Exper.	3.8 ± 0.38	5.34 ± 0.54	7.64 ± 0.42
		Theor.	4.2	6.1	10.1
	4	Exper.	5.2 ± 0.60	8.4 ± 1.02	11.9 ± 0.58
		Theor.	4.8	6.4	10.7
Preferential-Attachment	3	Exper.	4.4 ± 0.63	6.9 ± 0.86	9.1 ± 0.74
		Theor.	4.8	6.4	10.7
	4	Exper.	6.1 ± 1.00	9.5 ± 1.00	12.4 ± 0.71
		Theor.	5.2	6.8	11.3

ment provided by expansions with $L=2$ and for higher values of L. In the former case, the number of connections added to each specific node i increases directly with the square of the respective node degree, i.e., $\Delta E_2(i)=(1-p)k(k-1)/2$. Therefore, this expansion will tend to assign more edges to the nodes attached to the highest degree nodes, which also tend to have higher node degree, implying unbalance in the use of the ΔE reinforcement edges. The expansions implied by higher values of L will depend on the node degrees of the pair of nodes (i,j) at the extremities of the L pathway — see Fig. 2(b). More specifically, if those node degrees are k_i and k_j we have that $\Delta E_L(i,j)=p^{(L-2)}(1-p)k_ik_j$. Some insight can be achieved on such an effect by assuming the conservation rule $k_i+k_j=A$, from which we have that $\Delta E_L(i,j)=ck_i(A-k_i)$, for $c=p^{(L-2)}(1-p)$. Hence we have that the highest number of added edges $\Delta E_L(i,j)$ is obtained for $k_i=k_j$. At the same time, this type of expansion will assign fewer new edges whenever $k_i\neq k_j$, which tends to equalize the uniformity of the distribution of new nodes among a network grown by preferential attachment for the cases where k_i is substantially different from k_j.

As the network deconstruction is here assumed to occur by attacking the node with current highest degree, augmentation schemes that assign a large percentage of the ΔE reinforcement edges to a few nodes, such as is the case of the preferential attachment, are destined to perform poorly. In this sense, the random augmentation is expected to perform substantially better than the preferential-attachment scheme. On the other hand, given the tendency of Q augmentations to enhance connectivity around the nodes with highest degree associated with its equalizing effects, the potential of this strategy for enhancing the resilience to dominant node attack is likely not to be particularly effective.

V. RESULTS AND DISCUSSION

Several experimental investigations were performed by considering $N_0=50$, $i=1,2,3$ and $Q=3,4$. The first important point to be addressed regards the main global properties of the giant clusters obtained by the two different growth schemes, namely, random and preferential-attachment growth. Table I presents the average and standard deviation values of the total number of nodes, edges and average degree obtained for the two models considering $i=1,2,3$. The fact that the giant cluster appears near $i=1$ is clearly inferred by the relatively large standard deviations of the giant cluster sizes shown in the first column of Table I. The smaller dispersion of the values in that table for $i=2$ and 3 indicate that the giant clusters considered for augmentation had about the same global properties, allowing a relatively fair comparison of the resilience to attack. Table II shows the augmentation ratios ρ obtained experimentally and estimated from the data in Table I by using the equations developed in the previous section. The theoretical values are found to have provided a reasonably good prediction of the augmentation ratios, especially for the random strategy and for smaller values of i, as it would be expected from the assumptions adopted while developing the equations for ρ. It is also clear from Table II that the four augmentation tended to imply higher number of additional edges than the three augmentations, and that the values of ρ are reasonably close for the situations to be independently compared regarding attack resilience, namely $Q=3$ and 4.

The numbers of remaining nodes in the network under attack, after n removals of the nodes with current maximum degree (see Ref. [29]), are shown in Figs. 3 and 4, for random- and preferential-attachment initial networks, respectively. The growing models are identified by the curve marks (see respective captions). The effect of increasing values of i and, to a lesser extent of Q, on the resilience is promptly observed from these figures. In other words, the addition of ΔE edges, quantified by the augmentation ratios in Table II, contributed to substantially reinforcing the network structure and resilience to node attacks. The best resilience was observed for the situation involving initial networks adopting $i=3$ and $Q=4$ [see Figs. 3 and 4(f)], with the networking breakdown occurring only after $n/N_0>0.7$. As it could be expected given its performance in homogeneous networks [10], the random augmentation allowed the best resilience reinforcement in *all* situations. The Q augmentation presented the poorest performance, which was, however, superior to the preferential-attachment model at the very last stages of the attacks in several situations. Except for these cases, the preferential-attachment type of augmentation presented intermediate performance. Another interesting aspect is that the growing model chosen to produce the giant clus-

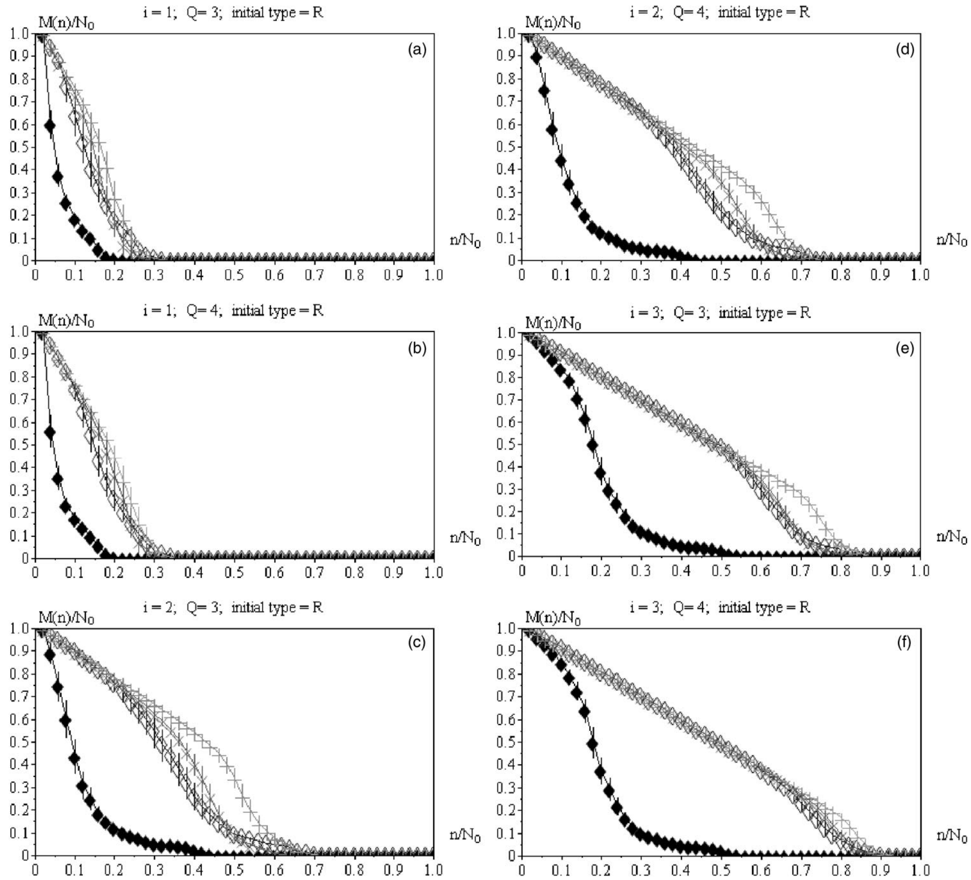

FIG. 3. The normalized number of nodes remaining in the giant cluster $[M(n)/N_o]$ in terms of the normalized number of removed nodes (n/N_o) for the cases (i)–(iii), $i=1,2$, and 3 and $Q=3,4$, where filled $\diamond$ = initial network, $+$= R, $\times$= PA, and $\diamond$ = QE indicate the augmentation model.

ters had little effect over the subsequent augmentations, which is supported by the similar curves in Figs. 3 and 4. This is possibly explained by the fact that the intense reinforcements implied by the Q augmentations tended to equalize the topological properties of the enlarged giant clusters. However, the preferential attachment did tend to perform slightly better for the situations involving giant clusters obtained through preferential attachment. Investigations with larger values of N_{GC} tended to produce similar results.

VI. CONCLUDING REMARKS

This paper described the investigation of the resilience of six hybrid network models obtained through the process of augmenting an initial connected network. This situation presents interest not only for its theoretical implications, but also because of practical concerns while trying to enhance the design of specific network systems in order to suit fault-tolerance specifications. The six network models included the random and preferential-attachment traditional networks augmented by random, preferential attachment and Q augmentations (for $Q=3$ and 4). Based on the concept of L expansions, the *Q augmentations* of existing networks was presented and investigated, including some statistical predictions of the resulting network properties, for the first time in this paper. Interestingly, the Q augmentations were found always to lead to unit clustering coefficient, a fact that partially clarifies the type of connectivity enhancement implied by such augmentations.

The obtained results revealed some interesting aspects which tended to agree with the statistical analysis. First, the augmentation of the initial giant cluster was observed to substantially enhance the resilience of the final network, at the expense of a larger number of edges. Second, the random model confirmed its superiority regarding the highest degree nodes attack, with the preferential-attachment networks coming second, except at the very last stages of the attack sequence, where the Q augmentations tended to provide behavior similar to that of the random networks. Another interesting result is the fact that the initial model (type and growth parameters) had little effect over the subsequent resilience enhancement obtained through the three considered augmentation schemes, except for the fact that the preferential-attachment augmentation tended to perform better when applied to giant clusters obtained by this same growing model.

The problem of reinforcement can be understood as a specific situation of a broader class of problems where one wants to *redesign* or *adapt* a given network in order to obtain specific topological or functional properties. Such a situation could arise in several contexts, for instance in internetwork-

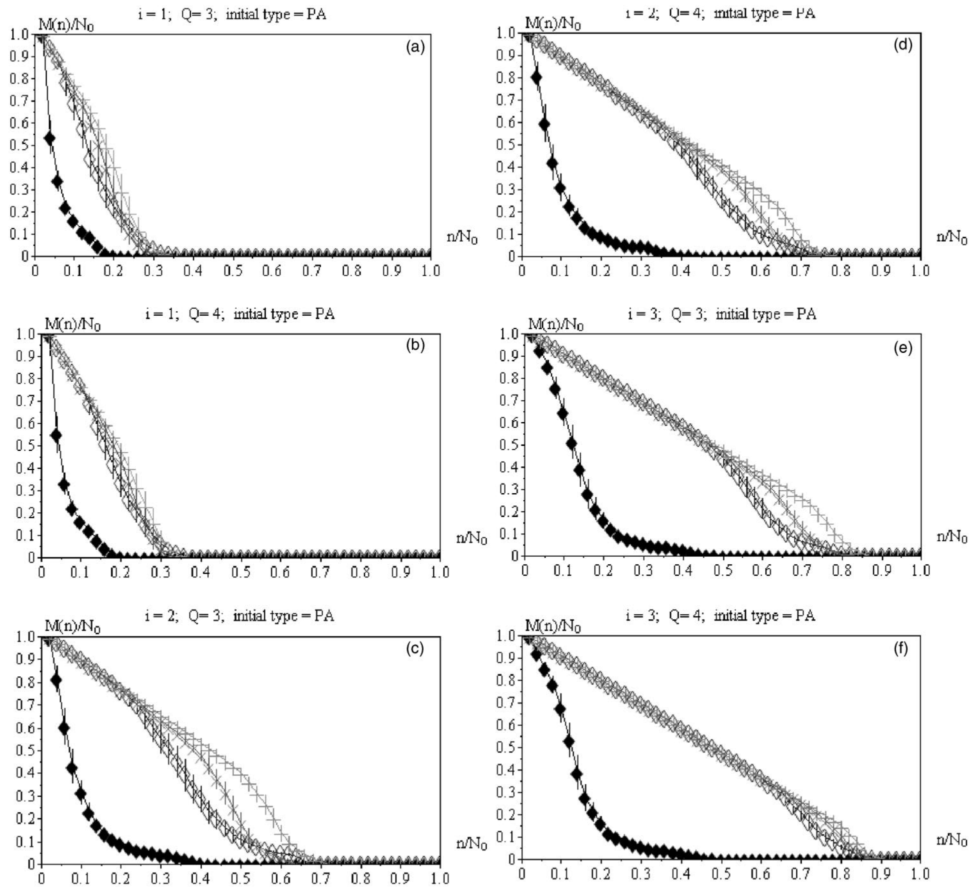

FIG. 4. The normalized number of nodes remaining in the giant cluster $[M(n)/N_o]$ in terms of the normalized number of removed nodes (n/N_o) for the cases (iv)–(vi), $i=1,2$, and 3 and $Q=3,4$, where filled ◇ = initial network, + = R, × = PA, and ◇ = QE indicate the augmentation model.

ing, electronic circuits (analogic or digital), and also biology. Regarding this latter situation, a particularly interesting case is the exposure of existing biological networks—including metabolic, protein, food chain, and ecological—to abrupt environmental variations of the geographical, environmental, and metereologic conditions that permeate the evolutionary process.

Future works may consider the evaluation of the performance of other hybrid systems, such as those obtained by union of two distinct models (e.g., Ref. [30]), progressive modification of the growing scheme (e.g., Ref. [31]), or even successive alternations of the augmentation schemes. Another interesting further work is to devise modifications of the Q-augmentation scheme where expansions are not applied indiscriminately over all nodes, but at random or selectively to specially critical nodes (e.g., those with low degree or betweeness centrality). Actually, such a line of reasoning ultimately leads to the following question: Given an existing network, how to identify the optimal augmentation scheme, i.e., that leading to the best overall resilience at the expense of the smallest number of additional edges?

Although this type of problem has been well developed in the context of traditional graphs (e.g., Ref. [24]), it would be interesting to revisit it by considering the new concepts and results from complex network research. Another related question is given an augmented network, how to identify the initial and/or expanding models?

For instance, the concept of L-conditional expansion [7] can be used to identify the regular connections implied by Q augmentations. In addition, it is worth observing that, in addition to enhancing the connectivity of the initial network, Q augmentation are also likely to promote higher regularity of node degree at the scale defined by Q. Possible means to identify augmentation schemes leading to high (or maximum) resilience is to use simulated annealing or the genetic algorithm.

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